On the Law of Human Mortality; and on Mr. Gompertz's new exposition of his Law of Mortality. By T. R. EDMONDS, B.A., *formerly of Trinity College, Cambridge.*

IN the last number of the *Assurance Magazine* (April, 1861), there appears a letter from Mr. Gompertz, in which reference is made to his and my claims to the discovery of the law, or part of the law, of human mortality. In this letter, Mr. Gompertz declares that my claim cannot interfere with his claim, because I had acknowledged the priority of his discovery. I admit the truth of this declaration, and can add thereto, that I also am able to declare, with equal truth, that his claim, as now described by himself, cannot interfere with any claim hitherto advanced by me. The law of human mortality, now claimed to be discovered by Mr. Gompertz, differs materially from the law which I have declared to exist. The two laws are in accordance with one another only at one period of life (extending, say, from the age of 15 to the age of 55 years), and then only in a partial degree. It is in this partial agreement, at this particular period of life, that the priority of Mr. Gompertz's discovery consists. The description of the law of human mortality, as *now* believed by Mr. Gompertz to exist, is contained in a paper, referred to in the letter above mentioned, and since published (page 454) in the Appendix to the *Report of Proceedings of the Fourth International Statistical Congress, held in London in the year* 1860.

The universal law of human mortality which I believe to exist (and have so believed for the last 33 years), may be described as consisting of three parts, all of great simplicity.

The first part is, that, from birth to extreme old age, human life is divided into three well-marked physiological periods, of growth, fruitfulness, and decay, which periods differ from one another in having distinctive rates of decrease or increase of their mortality, according to age. These periods may be described as those of infancy, fecundity, and senescence. The discovery of these three periods of different progressive mortality according to age, was first made public by Dr. Price, in the year 1769, as I have already stated in the *Assurance Magazine* (page 171) for October, 1860. According to Dr. Price, human life, from birth upwards, grows *gradually stronger* until the age of 10 years, then *slowly loses* strength until the age of 50, then *more rapidly loses* strength, until, at 70 or 75, it is brought back to all the weakness of the first month.

The second part of the law of human mortality is, that, from the beginning to the end of any one of the three stages of infancy, fecundity, or senescence, the rate of mortality varies with the age in a geometrical progression, the common ratio of which may, or may not, be different for different populations, or for the same population at different times. This law of increase or decrease is the simplest conceivable law. For example, suppose it to have been ascertained, by observation of the same population, that the annual mortality at the age 25 years was 1 per cent., and at the age 45 years, 1·80 per cent., and suppose it to be required to interpolate, according to some law, the mortality corresponding to intermediate ages. In that case, the assumption first made, as being the simplest, would most probably be, that the annual proportional increase according to age was constant throughout the entire period of 20 years of age. Adopting this assumption, it would ensue, that the common ratio of the geometric progression at which the mortality increases, is equal to the twentieth root of 1·80, or to 1·03. If every term is to the next preceding term as 1·03 to 1, then the twenty-first term will be to the first term as 1·80 to 1.

The third and last part of the law of human mortality consists in the permanency of each of the three common ratios of geometric progression of mortality, according to age, in all populations. The common ratios for the several periods of infancy, fecundity, and senescence, being known for one population are known for all other populations. The respective values of these three common ratios, for annual intervals of age, have been ascertained to be the following, nearly—viz., $\frac{1}{1\cdot479108}$, 1·0299117, and 1·0796923 (say $\frac{2}{3}$, 1·03, and 1·08), these being the numbers of which the common logarithms are $(-\cdot17)$, $(+\cdot0128)$, and $(+\cdot0333)$.

The public are indebted to Mr. Gompertz for the discovery (in 1825) of a portion of the second part of the law of human mortality. This portion is, that in one of the well-marked physiological periods of human life, the mortality increases with the age in a geometric progression, the common ratio of which may, or may not, be different for different populations. The period to which this discovery applies, is that of "fecundity," extending, say, from the age of 15 to the age of 55 years. The evidence offered of the existence of this portion of the law of human mortality was indirect. For Mr. Gompertz, without saying anything as to the division of human life into three, or any other number, of physiological stages or

periods, gave a general formula of geometric increase of mortality "*for a long period of man's life,*" without assigning any limits to such period. All the examples of the applicability of his formula, with one exception only, were founded upon observations of mortality made in the period of "fecundity." From these facts it may be fairly concluded, that Mr. Gompertz is entitled to the credit of the discovery, that in one of the three periods into which human life is divided, the mortality increases with the age in a geometrical progression, the common ratio of which may, or may not, be different for different populations.

In his paper, printed in the *Philosophical Transactions* of 1825, Mr. Gompertz expresses homely facts in transcendental language. Instead of supposing the mortality to increase in a geometrical progression with the age, he supposes the *power to oppose destruction* (which is inversely as the *mortality*) to decrease in equal proportions in equal infinitely small intervals of time. Instead of saying that the law of mortality, from the age of 15 to the age of 55 years, is such that the annual rate of mortality increases in a geometrical progression, of which the common ratio is (p), he states that the law is such that the survivors, or numbers left alive under the operation of this law for the term of (x) years, will be represented by the transcendental (dg^{p^x}). The law of mortality, however, is better and more simply expressed in terms of the mortality than it can be in terms of the survivors according to the same law. In the former case, we have (ap^x) to express the law; in the latter case we have the same law expressed by $y=10^{\frac{k^2a}{\lambda p}(1-p^x)}$, which is the quantity (dg^{p^x}), corrected and reduced to its simplest terms. Both expressions indicate the same facts, or the mathematical consequences of the same facts. In either case, where (a) and (p) are given, the only variable quantity is (x), measured from the time when the mortality was (a). It is not stated by Mr. Gompertz which of the two equivalent expressions for the law of mortality in the period of "fecundity" was first discovered by him. His expression for the survivors according to age, might have been discovered by inspection of published tables of the logarithms of survivors, at yearly intervals of age, accompanying ordinary tables of mortality. It might easily have been gathered from such inspection, that the differences between the logarithms of the survivors, at successive decennial intervals of age, are in geometric progression, and of the form (Cp^x). From this fact, the equation $y=dg^{p^x}$ might easily have been deduced.

In supplying evidence of the truth of his law of mortality, Mr. Gompertz invariably omits to compare the resulting mortality as deduced from his theoretical tables with the observed mortality according to age, which is or ought to be the foundation of such tables. In my own case, I have always supplied evidence of this complete and most simple kind. Nevertheless, Mr. Gompertz and his two officious, but accepted, advocates, concur in saying, that I have not given evidence of the truth of my theory of mortality, or of the existence of any law of mortality beyond that discovered by Mr. Gompertz, as applicable *"to a long period of man's life,"* to which no limits are assigned. This concurrence of views between Mr. Gompertz and his two advocates is the more remarkable, because it is the only instance of such concurrence to be found in the recently published papers of the three parties. His advocates advance claims on his behalf, which Mr. Gompertz has never advanced himself. The statements of the advocates are not supported by any corresponding statements of their principal. The only support which Mr. Gompertz gives his two advocates, is, to praise them for the talent which they have displayed in the advocacy of a claim rightfully or wrongfully put forward for his benefit. Mr. Gompertz appears to consider himself free from any responsibility in lending his approbation to papers containing statements, which, from his superior knowledge of the facts, he could not, with any regard to truth, have uttered himself. Except in the present instance, Mr. Gompertz has written nothing against my claim, or in support of the claim made on his behalf, which I am at all interested in contradicting. But with respect to his two advocates, I deny that either of them has said anything in depreciation of my claim which is true in substance or even in form.

Mr. Gompertz's theory of mortality (like my theory) rested greatly for support on the observations of mortality according to age of the populations of Sweden and Carlisle, made in the last century by Wargentin and Heysham respectively. The simplest evidence of the soundness of his theory would have been to show, that, taking successive decennial periods of age, the mortality deduced from either of his theoretical tables was in near agreement with the mortality at the same intervals of age observed by Wargentin or Heysham, and that both the theoretical and observed mortality were in geometric progression from the age of 15 to the age of 55 years. Instead of doing this, Mr. Gompertz has merely compared his series of survivors according to age with the series of survivors formed by Price and Milne from the observed facts. He

had previously taken no pains to inquire, whether the tables of Price and Milne were in agreement with the facts in mortality on which they are professed to be founded, and whether Price and Milne had not deviated from the facts presented, in order to smoothe irregularities of decrement, according to some ideal law of mortality which they may have entertained without expressing.

In proof of the soundness of my theory of mortality, I know no fact more convincing or interesting than one which has already appeared in the pages of the *Assurance Magazine* (year 1855, page 144). It is there shown, for that part of the total male population of England which was contained in five decennial periods of age, commencing at age 25 and ending at age 75 years, that the mortality observed in these decennial intervals of age was almost in exact accordance with my theory of mortality published in 1832—being three years before there existed any complete observation of the mortality of any part of the general population of England, except that of Carlisle, in the year 1787. To show the closeness of the coincidence to the present reader, the following extract is made from the more extended table, of which it forms part.

Annual Mortality per Cent., during 10 *Years ending with* 1850, *of all the Male Population of England comprised between the Ages of* 25 *and* 75 *Years.*

Between Ages..	25-35.	35-45.	45-55.	55-65.	65-75.
Fact observed....	·97	1·25	1·78	3·14	6·61
Theory	·96	1·28	1·75	3·22	6·78

In the Appendix (page 455) to the *Proceedings of the Fourth International Statistical Congress,* there is contained a statement by Mr. Gompertz of the view which he *now* entertains of the "*one continuous and uniform law of mortality from birth to at least the age of* 100 *years.*" He there states his belief that such law is expressed by the following equation, wherein (L_x) represents the numbers surviving or living at any age (x) measured from birth :—

$$L_x = \text{constant} \times A^{e^x} \times B^{e^x x} \times C^{q^x} \times D^{P_x}.$$

The following are the remarks made by Mr. Gompertz with respect to the applicability of the above formula:—"In all the tables I have examined, all the factors but C^{q^x}, being between the ages 20 and 60, are so nearly constant that the difference from it may be neglected. This equation may be put into the form,

constant $\times C^{q^x}$." Afterwards, he goes on to say that, by the disappearance of the two first transcendental factors, the equation from the age 60, and ever after, is of the form $L_x = \text{constant} \times C^{q^x} \times D^{P_x}$.

From the above, it will be manifest that it is only at the interval of ages from 20 to 60 years that there is any agreement between Mr. Gompertz's theory and my theory of mortality; for, according to my theory of mortality, in either of the three stages into which human life is divided, the equation for the living or surviving at any age (x), from the beginning of that stage, is $L_x = \text{constant} \times g^{p^x}$, the quantity ($p$) having a fixed and distinct value in each of the three stages; whilst, according to Mr. Gompertz's theory of mortality, human life is divided into four stages, and in three of the four stages no more than two out of the four transcendentals which enter into the above general formula of mortality ever disappear.

Mr. Gompertz also gives the following as his general equation of mortality for the whole of life expressed in common logarithms—

$$\lambda L_x = \text{constant} + ke^x + k_1 e_1^x x - nq^x - P_x,$$

and makes the following observations thereon :—

"ke^x is at birth $= k$ and decreases regularly as x increases, and becomes, before the age of 20, and ever after, in the table I have examined, perfectly insignificant.

"$k_1 e_1^x$ is of no value at birth, but increases in a very short time to a maximum of significant value; but in less than 12 months, in the table I have examined, decreases to perfect insignificance.

"nq^x is $= n$ at birth, and continually increases, with the increase of x, to the remotest age.

"P_x is perfectly insignificant at birth, and till an age but a few years below 60, and it then continually increases with the age until far beyond the age of 100, and then decreases to perfect insignificance."

In speaking of the equation $L_x = AB^{q^x}$, used in his paper of 1825, Mr. Gompertz remarks—"That AB and q were supposed to represent constant quantities—or, at least, were shown to differ very little from constants—for a very long term of years (for instance, about 50 years), but differing a little for length of term and from one locality to another; and A and B and q being so related to one another, that, supposing them to be constant, we should have $A \times B = L_0$, the number at birth. But, in making the investigation, I did not pretend that A and B were absolutely constant. They were determined from a random selection from three distant periods of age, from a statement of the number of

persons who will be living at different ages out of a certain number of persons stated to have been born."

In neither of the two papers referred to as recently published in the name of Mr. Gompertz do I find any statement relative to our respective claims which impugns the correctness of any statement made in my *Life Tables*, or in my paper contained in the *Assurance Magazine* for October, 1860. I do not believe that Mr. Gompertz has ever complained of my having failed duly to acknowledge the share which he had in the discovery of the law of human mortality. Since the publication, in 1832, of my *Life Tables*, I have, at various times, been in free personal communication with gentlemen who had been, a short time previously, in free personal communication with Mr. Gompertz. From none of these gentlemen have I heard that Mr. Gompertz imagined himself to have been wronged by me. As Mr. Gompertz himself, during the space of 28 years, remained insensible to the supposed wrong, it may fairly be presumed that such supposed wrong had no existence, and that the charge of wrong brought against me by a third party (Mr. De Morgan) is entirely without foundation.

In the *Assurance Magazine* for July, 1860, Mr. De Morgan brought the charge against me of having, in the year 1832, in my *Life Tables*, "unfairly suppressed due acknowledgment to the writings of Mr. Benjamin Gompertz." After reading my paper in the next following Number of the *Magazine*, Mr. De Morgan writes as follows (page 214) in the Number for January, 1861:— "If Mr. Edmonds had given all the description which he has now given . . . . . . there would have been no suppression." This admission appears to me to amount to the virtual abandonment of his original charge, although Mr. De Morgan denies this by saying that suppression existed in 1832 and did not cease until the year 1860. The erroneous conclusion of Mr. De Morgan is founded on the erroneous assumption, that, in 1860, I made any statement or admission more favourable to Mr. Gompertz's alleged claims than I had made in 1832. I do not think that Mr. De Morgan will find anyone (even Mr. Gompertz himself) to concur with him in such an erroneous assumption. In the year 1832 I wrote as follows:— "The honour of first discovering that some connexion existed between tables of mortality and the algebraic expression (a^{b^x}) belongs to Mr. Gompertz." These words appear to me to convey a sufficient intimation of the high value I then attached to the discovery, and of the credit due to Mr. Gompertz for making it. The same words conveyed a suggestion to the reader to inquire and judge for himself of

the comparative merits of my complete law and Mr. Gompertz's then incomplete law of mortality. The paper which I wrote in the *Assurance Magazine* of October, 1860, was written in self-defence (against an outrageous attack of Mr. De Morgan), for the purpose of circumscribing, within just and narrow limits, the favourable admission which I had made in general terms in the year 1832.

In a question like the present, when I have to meet an apparently unfounded accusation, it appears to me to be of importance that the reader should know that Mr. De Morgan and myself were not strangers to one another in our years of student life; that previous to publication, in 1832, my *Life Tables* obtained his approval; and that, without giving me any intimation of the withdrawal of this approval, he had been secretly writing and speaking against me (in relation to Mr. Gompertz's supposed claim), for the space of 28 years. Mr. De Morgan and myself were students of the same year of admission, of the same University, of the same College, and of the same class-room of the College. We attended the same mathematical class for three years—or, rather, we should have done so if I had not omitted to read mathematics during the whole of the second year. In our third year the present Astronomer-Royal was the mathematical lecturer of our class, which, near the termination of the year, was reduced to two students—Mr. De Morgan and myself. In the year 1832, before publication, I sent Mr. De Morgan a printed copy of my *Life Tables.* A few days afterwards, at a personal interview, he expressed his general approbation of the work, but objected to one sentence only, which had no connexion with Mr. Gompertz or his supposed claim. He recommended me to have this sentence (and the page containing it) cancelled. This I declined doing; and since that time I have had no communication with Mr. De Morgan, except on one occasion, when I discovered that he had ceased to be favourably disposed towards me—for no reason, that I could imagine, except my refusal to act upon his recommendation given in 1832. It appears to me that Mr. De Morgan, after expressing to me a favourable opinion of my *Life Tables* in 1832, was not justified in changing that opinion, and acting upon that changed opinion, without previously communicating with me, and calling for an explanation. If he had done so, I should have offered the explanation which is given in the *Assurance Magazine* of October, 1860—which explanation, as Mr. De Morgan himself admits, would have exonerated me from the charge of "unfair suppression" if it had been given in the year 1832.

The formula by which Mr. Gompertz represents the number surviving according to age in one of his four periods of human life is $y = dg^{p^x}$. This formula is defective in several respects. In the correct formula the exponent of (g) is $(p^x - 1)$, and not (p^x) as stated by Mr. Gompertz. The factor (d), in the formula of Mr. Gompertz, is superfluous; and, besides being superfluous, is defective as being a composite quantity, of which the other factor (g) forms part. I will proceed to place before the reader the successive steps by which the correct formula was obtained by me 33 years ago; and I will afterwards exhibit the course of investigation pursued, or intended to be pursued, by Mr. Gompertz in arriving at his defective formula, giving Mr. Gompertz the benefit of the most recent corrections made by himself and his two advocates.

The problem for solution is this:—Given the annual mortality (a) at the beginning of a period (x) years of age, given also the common ratio (p) of annual geometrical progression of mortality according to age; required to find an expression for (L_x), or the number living and surviving at the end of the (x) years of age, in terms of (a), (p), and other known quantities.

Previous to forming the differential equation, it is essential that the nature of the quantity (a), given to represent the proportional mortality for one year when the mortality is constant, should be determined. There are two different quantities by which this annual mortality may be expressed. In one case, (a) is the finite decrement for one year, and is such that the number living at the beginning of the year of age is to the number surviving at the end of the same year as $(1 + a)$ is to 1. In the other case, the annual mortality at the age 0 is represented by a quantity α, which is the product of an indefinitely small decrement $\left(\frac{\alpha}{\nu}\right)$ multiplied by an indefinitely large number (ν). If the finite decrement for one year, when the mortality is constant, is known to be equal to (a), the value of (α) when $x = 0$ may be determined from the equation $e^\alpha = 1 + a$, which gives $\alpha =$ hyperbolic logarithm of $(1 + a)$. If $a = {\cdot}0063845$, then will $\alpha = {\cdot}0063643$.

In Mr. Gompertz's investigations no statement is made of any algebraical or arithmetical value of (a) or (α), although the object of his investigation was to find an expression for the number living at any age (x) in terms of the annual mortality, whether (a) or (α). This omission of separate notice of the most important quantity in the formula affords ground for doubting whether Mr. Gompertz was acquainted with the nature of the quantity used by him to

denote the annual mortality when (x) was equal to 0. An independent table of mortality could not have been constructed from Mr. Gompertz's formula without the previous knowledge now indicated. It so happens that Mr. Gompertz has published no portion of a theoretical table of mortality which is independent of previously constructed tables. His examples show how interpolations may be made in existing tables—not how portions of new tables may be constructed from the mortality only. The differential equation of Mr. Gompertz is erroneous if he has used the finite decrement (a) to represent the mortality; for the differential equation is true for all values of (x) (whole number or fractional) only when the annual mortality is represented by (α), which is equal to the hyp. logarithm of $(1+a)$.

Beginning with the differential equation, we have—

$$dy = -y\alpha p^x dx;$$

integrating,

$$\log. y = \log. d - \frac{\alpha p^x}{\log. p};$$

when $x=0$,

$$\log. \mathrm{L}_0 = \log. d - \frac{\alpha p^0}{\log. p};$$

by subtraction,

$$\log. \frac{y}{\mathrm{L}_0} = \frac{\alpha}{\log. p} - \frac{\alpha p^x}{\log. p} = \frac{\alpha}{\log. p}(1-p^x);$$

whence

$$\frac{y}{\mathrm{L}_0} = e^{\frac{\alpha}{\log. p}(1-p^x)} = 10^{\frac{k^2\alpha}{\lambda p}(1-p^x)} = 10^{c(1-p^x)} = g^{1-p^x};$$

whence

$$y = \mathrm{L}_0 \times g^{1-p^x} = \mathrm{L}_0 \times \frac{g}{g^{p^x}} = \mathrm{L}_0 \times \frac{1}{g^{p^x-1}}.$$

Instead of the correct expression for (y) or (L_x), just obtained, Mr. Gompertz gives the defective equation $y=dg^{p^x}$. In deducing the latter erroneous value, Mr. Gompertz stops at the second step of the investigation, and thus fails to discover that the constant (d) is a compound quantity, including (g) as one of its factors. According to the defective process of Mr. Gompertz—

$$\log. \frac{y}{d} = -\frac{\alpha p^x}{\log. p}, \text{ so that } \frac{y}{d} = e^{-\frac{\alpha p^x}{\log. p}} = 10^{-\frac{k^2\alpha}{\lambda p}p^x};$$

and, putting

$$+c = \left(-\frac{k^2\alpha}{\lambda p}\right), \text{ and } 10^{+c} = g,$$

he gets

$$y = d \times 10^{-\frac{k^2\alpha}{\lambda p}p^x} = d \times 10^{+cp^x} = dg^{p^x}.$$

The latter equation is defective by reason of the constant (d) not having been determined by the aid of the equation $\log. L_0 = \log. d - \frac{a}{\log. p}$. If it had been so determined, (d) would have been found to be equal to $L_0 \times \frac{1}{g}$. If he had made the proper substitution, his equation would have become

$$y = L_0 \times \frac{1}{g} \times g^{p^x} = L_0 \times g^{p^x - 1}.$$

In addition to the defect just mentioned, there is the further defect of putting (g), whose exponent is positive, equal to $10^{-\frac{k^2 a}{\lambda p}}$, a quantity greater than unity whose exponent is negative, and thus transferring to the numerator a quantity which originally formed the denominator of a fraction. If Mr. Gompertz had not made this unnecessary change in the sign of the exponent of (g), his corrected formula, on substituting g for $\frac{1}{g}$, would have become

$$y = L_0 \times \frac{1}{g^{p^x - 1}}.$$

In addition to the defect last mentioned in Mr. Gompertz's formula, there is the further defect of using in the formula any such quantity as (L_0) at all; for (L_0) is the common multiple of all the terms of the series for the living at successive ages, and conveys no information whatever as to the proportions surviving at different ages, which is all the information sought by the formula. If, for (L_0), we substitute unity, we shall have the equation complete in its most simple terms and form, viz.:—

$$y = \frac{1}{g^{p^x - 1}} = g^{1 - p^x} = 10^{c(1 - p^x)} = 10^{\frac{k^2 a}{\lambda p}(1 - p^x)},$$

the last being the form in which it was published by me in the year 1832.

In my paper of October, 1860, I designated the quantity (d) in Mr. Gompertz's formula as superfluous, useless, and indeterminate. The new advocate of Mr. Gompertz objects to my use of the word "indeterminate," because the word is usually limited to the description of a quantity which cannot be determined. It appears to me, that my use of the term is in accordance with his own definition; for Mr. Gompertz and his two advocates have not hitherto been able to determine the value of (d) by decomposing it into its two constituent factors $\left(L_0 \text{ and } \frac{1}{g}\right)$, by doing which, they might have

simplified aud rectified the original defective equation $y=dg^{p^x}$. Moreover, I would submit, that a quantity, whether intrinsically defective, as (d), or admitting an endless variety of arbitrary values, as (L_0) (none of which concern the object of the investigation), may with reason be called "indeterminate," because there is nothing to be determined, or worthy of being determined.

In the same paper I also termed (d) a particular value of (y), whilst, in strictness, I should have said that that part of (d) which did not contain (g) is a "particular value" of (y). I had previously said, that (L_0), or y when $x=0$, was equal to $\frac{d}{g}$, which is the same thing as saying that (d) was a particular value of (gy). In order to avoid circumlocution, I omitted to say that the argument was the same—whether (d) was a particular value of (y), or of (gy). On account of this omission, the new advocate of Mr. Gompertz has charged me with having committed "a far more serious error" than that of calling (d) an indeterminate constant. He says nothing of the really serious error of Mr. De Morgan, in the exposing of which I committed the pretended verbal error. In the *Assurance Magazine* of July, 1860, page 88, Mr. De Morgan gives a formula, designed to express the number living at any age in terms of the mortality, in which formula he omitted the most essential part, the quantity (a), representing the mortality. It may be said, on Mr. De Morgan's behalf, that he copied the error previously committed by Mr. Gompertz. On the other hand, it is to be stated, that Mr. De Morgan reproduced the investigation and formula of Mr. Gompertz, in order to prove that they were as good as mine, designed for a similar purpose. Having regard to the task voluntarily undertaken, it is clear that Mr. De Morgan was responsible for the error adopted, whether originally committed by Mr. Gompertz or his printer.

Mr. Gompertz has recently presented to the Institute of Actuaries a corrected copy of his paper, which was printed in the *Philosophical Transactions* of 1825. The corrections with which we are now concerned, relate to errors in the printed investigation of his formula $y=dg^{p^x}$, and especially to two admitted errors repeated—abq^x and abq^x, immediately following one another in two separate equations. The correction of the first error, abq^x, is stated to be $a\dot{x}q^x$, or $adxq^x$; and of the second error, abq^x, it is stated to be aq^x. I have had the opportunity of minutely examining the printed errors and their corrections, and do not concur, with the new advocate of Mr. Gompertz, in the opinion that either of the two

errors has its source in a mistake of the printer, (b) being substituted for ($\times$), the sign of multiplication. The first of the two corrections may be passed over as genuine and unobjectionable, although the (dx) occupies an unusual position, coming before, instead of after, (q^x). With regard to the second correction, I question its admissibility. For the equation, as corrected, is more defective, in form, at least, than was the equation intended to be corrected. A necessary result of the integration made, was the appearance of a new constant, (b) or (d), on one side or the other of the equation. The correction now made would leave no new constant on either side of the equation. In the absence of any satisfactory account of the second of the two admitted errors, I will now offer, for consideration, the explanation of this error which occurred to my mind before Mr. De Morgan gave it as his opinion that (b) was a constant, additional to, and independent of, the constant (d)—both constants being introduced (rightly or not) by Mr. Gompertz in his process of integration. The source of Mr. Gompertz's error, in my belief, was, that he had carelessly and erroneously concluded, that the constant (b) on one side of the equation, was the equivalent of the constant (d) on the other side of the equation.

In the same paper, I stated that the unknown constant (b) had been introduced by Mr. Gompertz through his incorrect process of integration; and that such constant was not only superfluous and useless (as Mr. De Morgan had declared it to be), but that it was erroneous also. I supported this statement by giving a new proof of my formula $y=10^{\frac{k^2a}{\lambda p}(1-p^x)}$, and showed, without the aid of the differential calculus, that no quantity similar to (b) formed part of the formula. The new advocate of Mr. Gompertz does not call in question the truth of the substantial part of my statement, but accuses me of having committed an error in describing the mode in which the error was supposed to have been arrived at by Mr. Gompertz. He says, that the process of integration had not commenced when the erroneous (b) first appeared. In reply, I have to state, that there are two distinct errors of (b), both appearing after the completion of the differential equation. I have also to state, that the first error only has been satisfactorily corrected by the substitution of ($\dot{x}$), or dx, for (b); and the second erroneous (b) is, apparently, not a printer's error, but a new and erroneous constant, introduced by Mr. Gompertz immediately after the disappearance of (dy) and (dx) from the differential equation.

It is usual to consider the process of integration as commencing at the step immediately following the completion of the differential equation. In the present case, I will venture to offer the opinion, that the completion of the differential equation was the first and most important step in Mr. Gompertz's process of integration. For the integral sought was known, or believed to be known, by Mr. Gompertz, before the differential was known to him, as I have already shown, at page 176, in the paper before referred to. To know the correct differential and the correct integral, generally includes the knowledge of all the intermediate steps. It is doubtful, however, whether Mr. Gompertz possessed any exact knowledge either of the integral or differential sought. For his integral is erroneous, in exhibiting (p^x) as the exponent of the factor (g); and his differential equation would be erroneous, if the quantity (a) is intended to represent the finite annual decrement when the mortality is constant.

In the paper before referred to, I gave, as a quotation from Mr. Gompertz's paper in the *Philosophical Transactions* of 1825, a passage which has been repeatedly quoted as the essential part of his law of mortality. In doing this, however, I omitted the first word of the sentence, which is "if;" which word is used by Mr. Gompertz to signify "let it be assumed." In consequence of the omission of this insignificant word, I have been charged by the new advocate of Mr. Gompertz with quoting a sentence or passage which "has no existence in the writings of Mr. Gompertz." The writer does not describe the passage misquoted, although he refers to its place in the *Assurance Magazine* of October, 1860, nor does he state in what the alleged misquotation consisted. If he had done so, the charge (made in italics) would have appeared too ridiculous to be seriously entertained. I can truly say of this new advocate of Mr. Gompertz, whilst using his own expressive language, that his remarks "contain errors equally grave with those falsely imputed" to me.

The new advocate of Mr. Gompertz alleges, in opposition to the most obvious facts, that the defective quantity (d) "is really an essential part of the formula, and, *as such, is employed by Mr. Edmonds himself.*" There is not the remotest approach to truth in this statement, enforced by *italics*. The defective quantity (d) is obtainable from the equation $d = \frac{L_0}{g}$, and, when obtained, is worth less than nothing. Instead of attempting to prove that I had ever made use of such a quantity as (d), the new advocate of

Mr. Gompertz attempts to prove that I had used the correct but superfluous quantity (L_0) as distinct from unity. This he entirely fails to do, for he reproduces the identical figures which were given by me as obtained by the substitution of unity for L_0.

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*On the various methods pursued in the Distribution of Surplus among the Assured in a Life Assurance Company; with a comparison of the relative merits of such methods.** *By* WM. POLLARD PATTISON, ESQ., *of the London and Provincial Law Assurance Society.*

[The Author of this Essay obtained for it the Prize offered by the Institute for an Essay on the subject.—ED. *A. M.*]

THERE is, probably, no department of an actuary's pursuits where there is such diversity of practice as that of which this paper treats. After the calculation of the premiums and the periodical valuations, the distribution of surplus is infinitely the most important duty that an actuary has to perform. The premium income of the various Life Offices now exceeds £8,000,000 a year, and probably not less than £1,000,000 is annually divided as surplus. Within the last 14 years, one Office alone has divided upwards of £1,200,000; and another, for many years past, has annually distributed as surplus more than £100,000. This will be sufficient to show the magnitude of the interests concerned; and yet, as will be seen in the sequel, these distributions are, in some cases, made in a most arbitrary manner—without reference to principles of justice and equity, and without the basis of accurate reasoning. After what has been written on this subject by Mr. Jellicoe and Professor De Morgan, it is not to be supposed that any actuary would defend the schemes here referred to as equitable; those who *do* follow them, content themselves with stating their advantages. In many cases, however, their conservatism arises from the difficulty of disturbing vested interests; and the maxim, "*Omnis innovatio plus novitate perturbat quam utilitate prodest,*" is very attractive to those in office, upon whom would fall the labour of carrying out a new system. Hence, in this, as in many other cases of old-established errors, all true reform must come from without. But the application of prin-

* By the title of this paper it will be perceived that the methods actually pursued are those which are the subject of investigation and comparison. I emphatically wish it to be understood that they are in no sense theoretical, but such as are adopted at the present time, and mostly, too, by Offices of high standing. The names have been carefully excluded, and everything that might lead to their identification.